Approximate Nearest Line Search in High Dimensions

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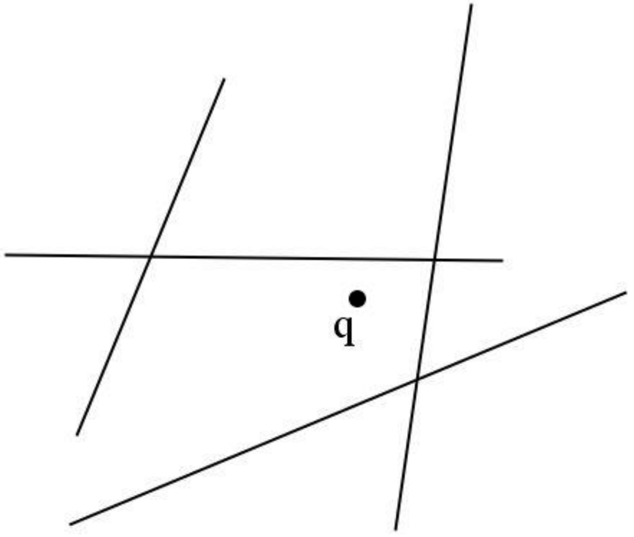
The NLS Problem

- Given: a set of N lines L in $\mathbb{R}^d$



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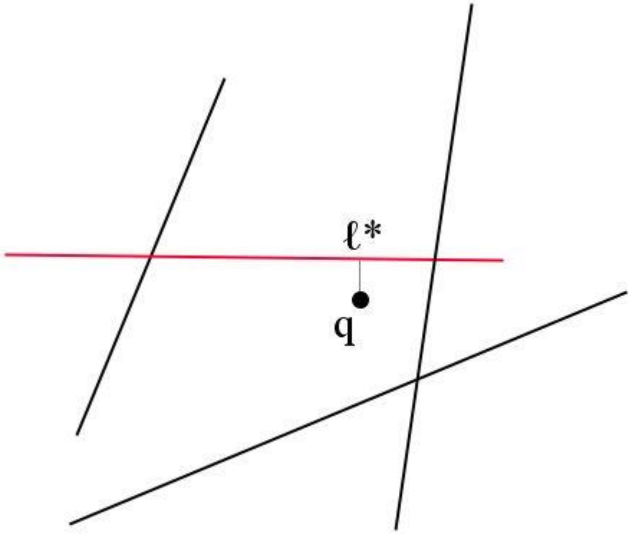

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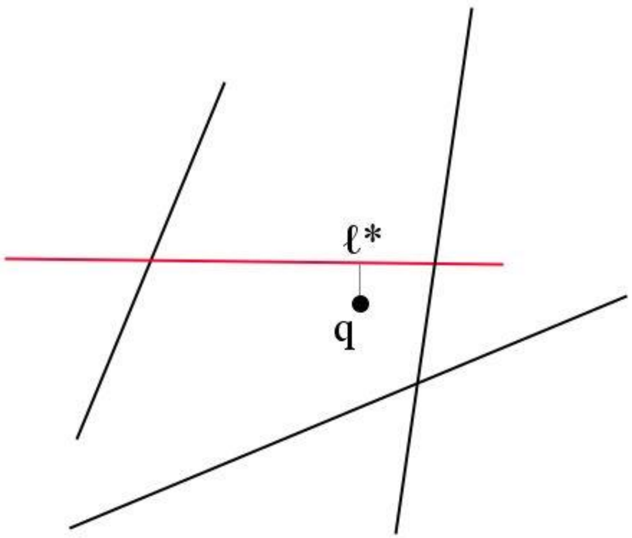

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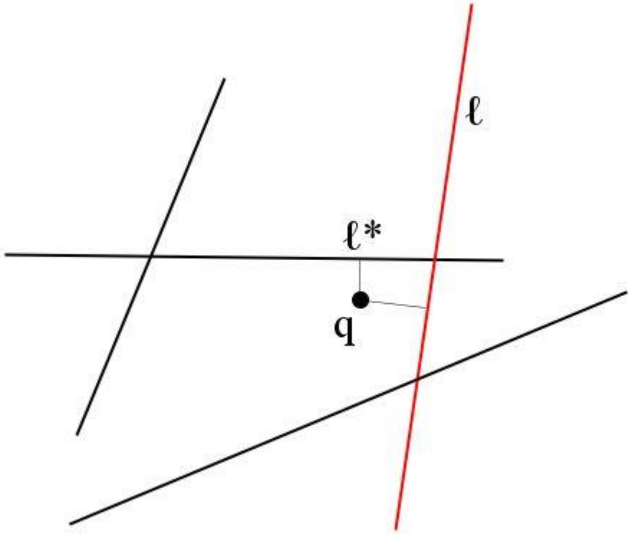


Approximation

- Finds an approximate closest line ℓ

$$dist(q,\ell) \leq dist(q,\ell^*)(1+\epsilon)$$

Nearest Neighbor Problems

Motivation

Previous Work

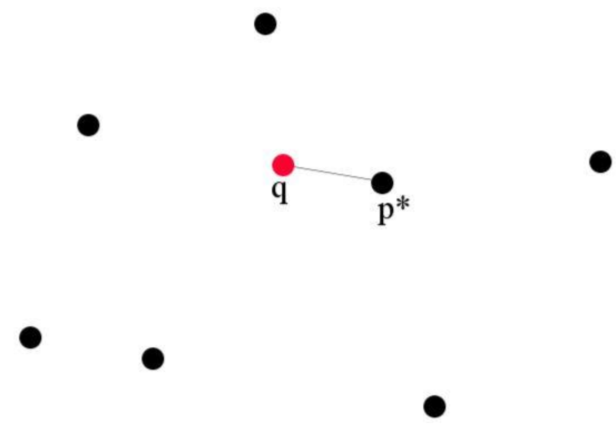
Our result

Notation

BACKGROUND

Nearest Neighbor Problem

NN: Given a set of N points P, build a data structure s.t. given a query point q, finds the closest point p^* to q.



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- Applications: database, information retrieval, pattern recognition, computer vision
  - Features: dimensions
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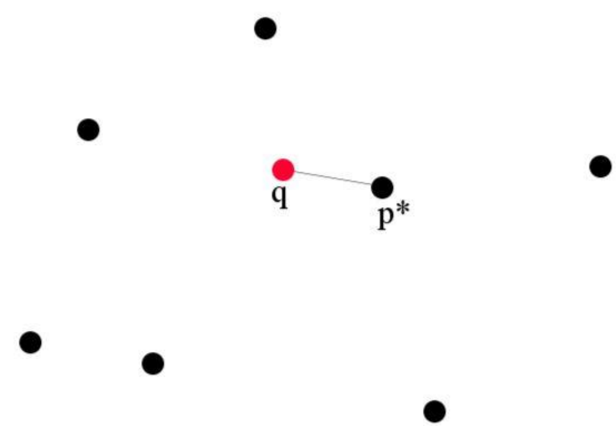

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- Applications: database, information retrieval, pattern recognition, computer vision
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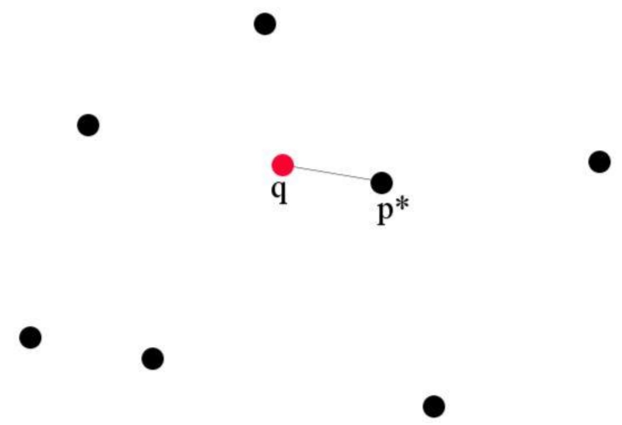
- Current solutions suffer from "curse of dimensionality":
  - Either **space** or **query time** is **exponential** in d
  - Little improvement over linear search



Approximate Nearest Neighbor(ANN)

- ANN: Given a set of N points P, build a data structure s.t. given a query point q, finds an approximate closest point p to q, i.e.,

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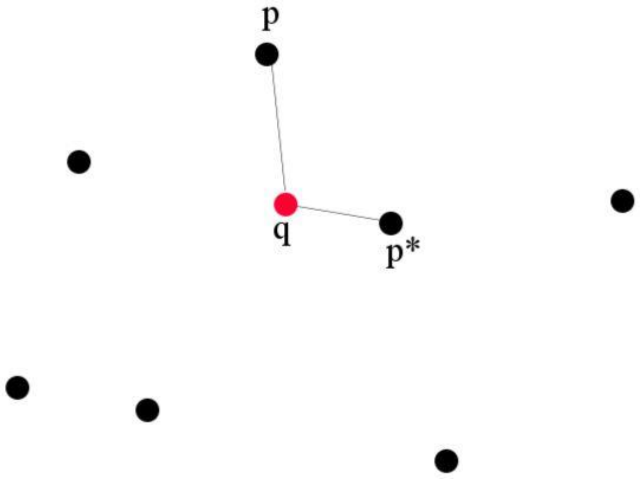

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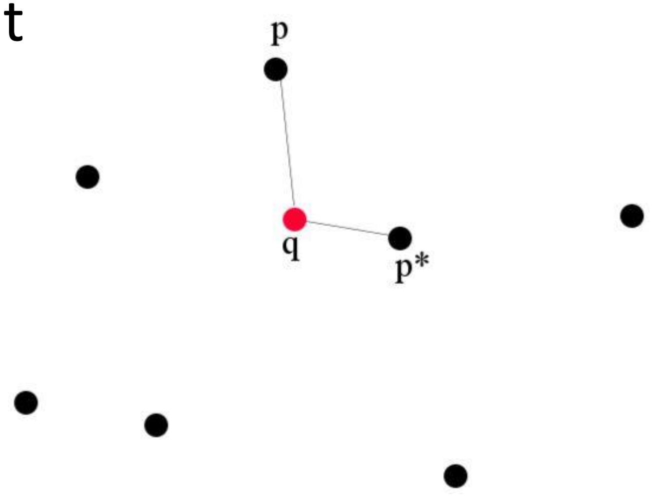
- There exist data structures with different tradeoffs. Example:
  - Space: $(dN)^{O\left(\frac{1}{\epsilon^2}\right)}$
  - Query time: $\left(\frac{d \log N}{\epsilon}\right)^{O(1)}$



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One of the simplest generalizations of ANN: data items are represented by k-flats (affine subspace) instead of points

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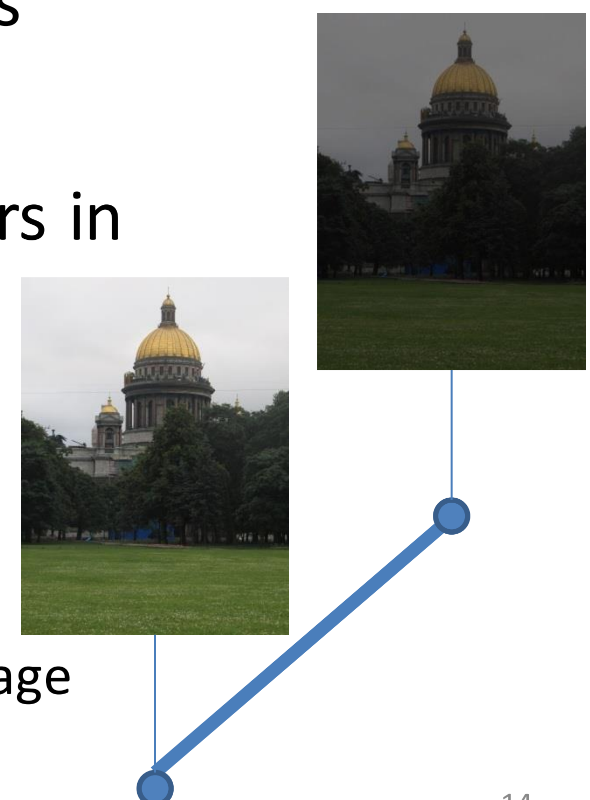
- Model data under linear variations
- Unknown or unimportant parameters in database
- Example:
  - Varying light gain parameter of images
  - Each image/point becomes a line
  - Search for the closest line to the query image



Previous and Related Work

- Magen[02]: Nearest Subspace Search
  - Query time is fast : $\left(d + \log N + \frac{1}{\epsilon}\right)^{O(1)}$
  - Space is super-polynomial : $2^{(\log N)^{O(1)}}$

Previous and Related Work

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Dual Problem: Database is a set of points, query is a k-flat

- [AIKN] for 1-flat: for any $t > 0$
  - Query time: $O(d^3 N^{0.5+t})$
  - Space: $d^2 N^{O\left(\frac{1}{\epsilon^2} + \frac{1}{t^2}\right)}$

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  - Query time: $O(d^3 N^{0.5+t})$
  - Space: $d^2 N^{O\left(\frac{1}{\epsilon^2}+\frac{1}{t^2}\right)}$
- Very recently [MNSS] extended it for k-flats
  - Query time $O\left(n^{\frac{k}{k+1-\rho}+t}\right)$
  - Space: $O(n^{1+\frac{\sigma k}{k+1-\rho}} + n \log^{O\left(\frac{1}{t}\right)} n)$

Our Result

We give a randomized algorithm that for any sufficiently small ϵ reports a $(1+\epsilon)$-approximate solution with high probability

- Space: $(N+d)^{O\left(\frac{1}{\epsilon^2}\right)}$
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- Only uses reductions to ANN

Notation

- L : the set of lines with size N
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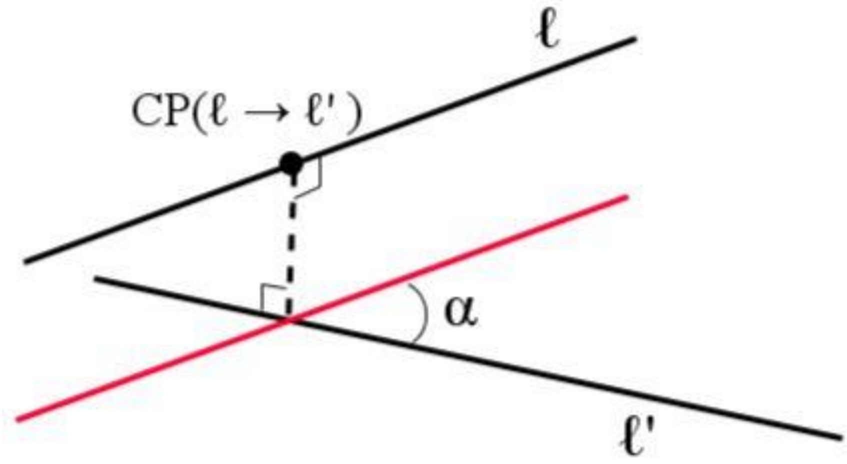

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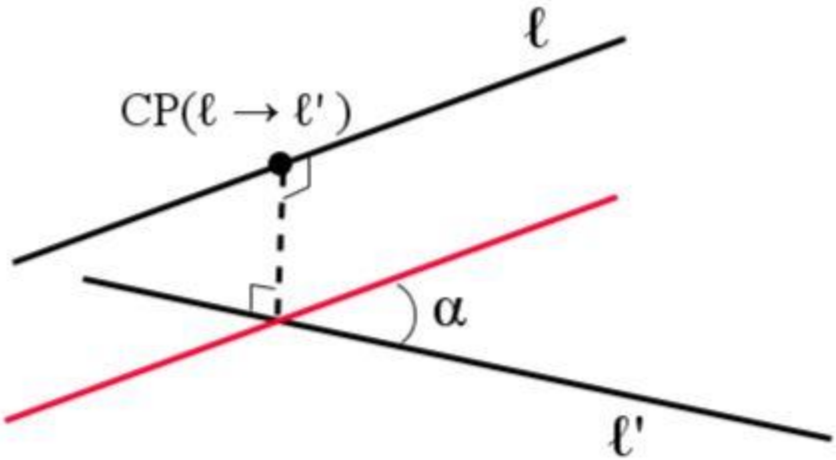
- δ-close: two lines ℓ , ℓ' are δ-close if $\sin(angle(\ell, \ell')) \leq \delta$. Similarly we define δ-far/ strictly δ-close/ strictly δ-far



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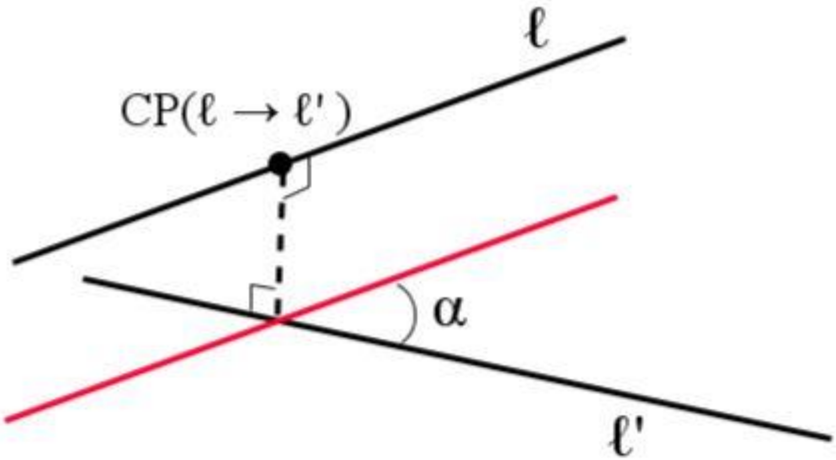
- $CP_{\ell_1 \to \ell_2}$: closest point on ℓ_1 to ℓ_2



Unbounded Module

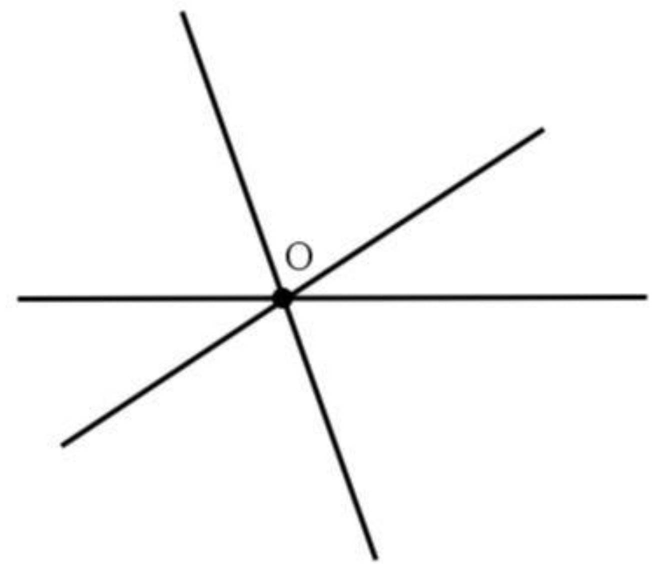
Net Module

Parallel Module

MODULES

Unbounded Module - Intuition

- All lines in L pass through the origin O



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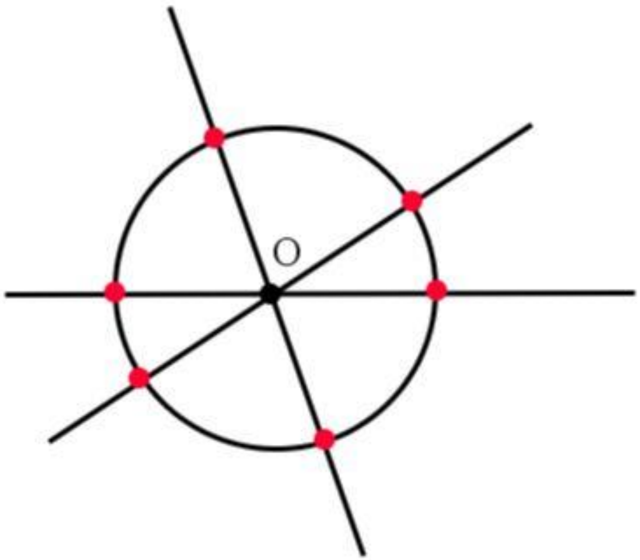
  - Project all lines onto any sphere $S(o, r)$ to get point set P
  - Build ANN data structure $ANN(P, \epsilon)$



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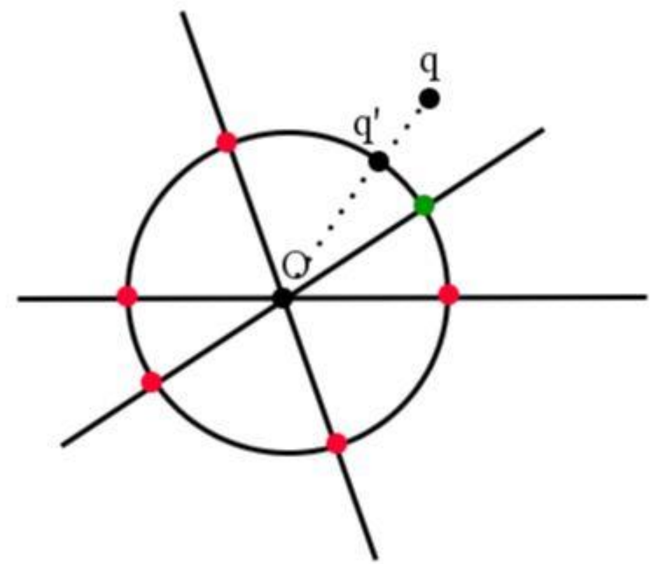
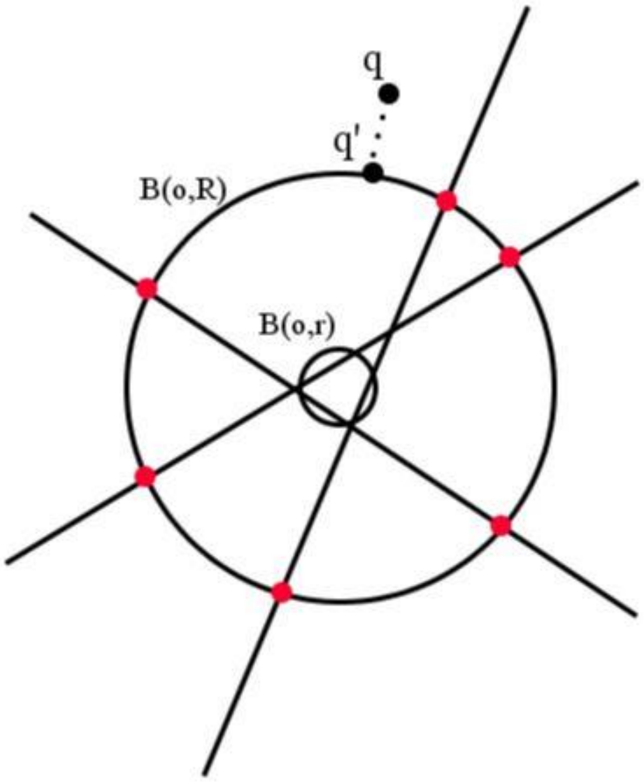


Unbounded Module

- All lines in L pass through a small ball $B(o,r)$
- Query is far enough, outside of $B(o,R)$
- Use the same data structure and query algorithm



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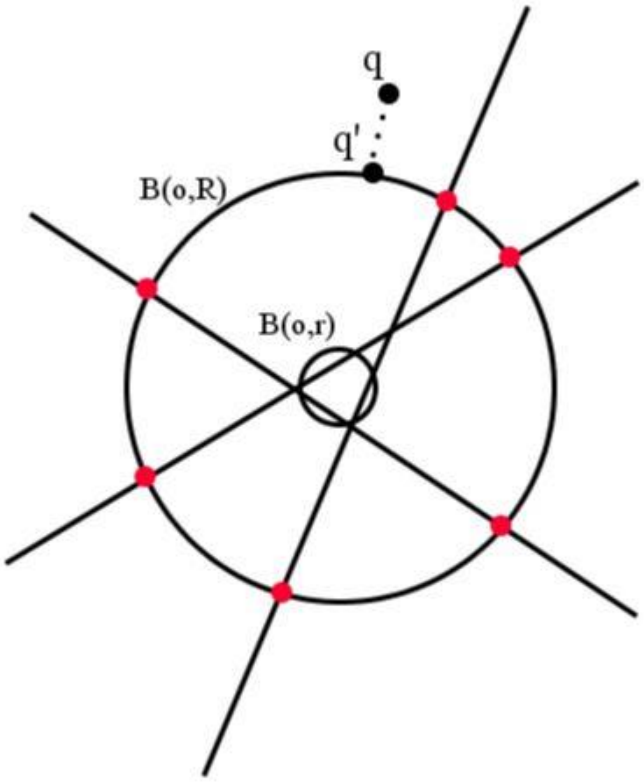

Lemma: if $R \geq \frac{r}{\epsilon\delta}$, the returned line ℓ_p is

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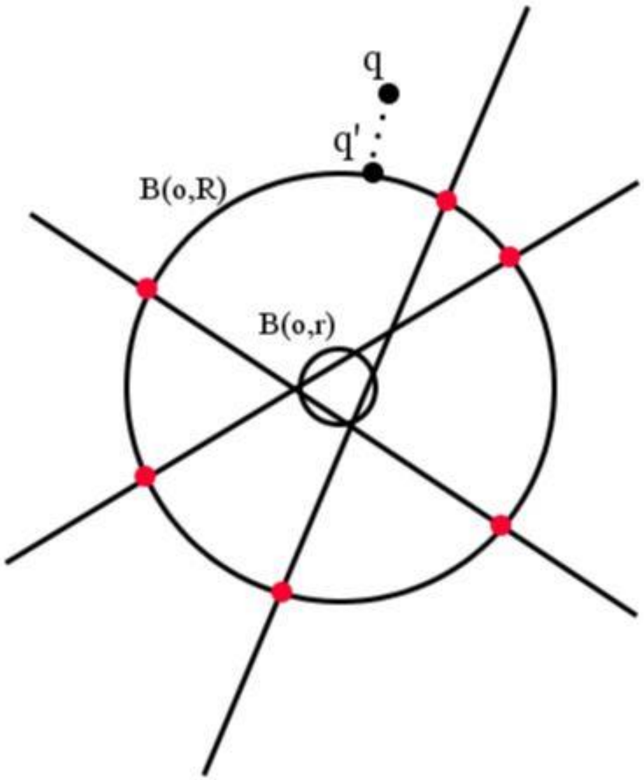

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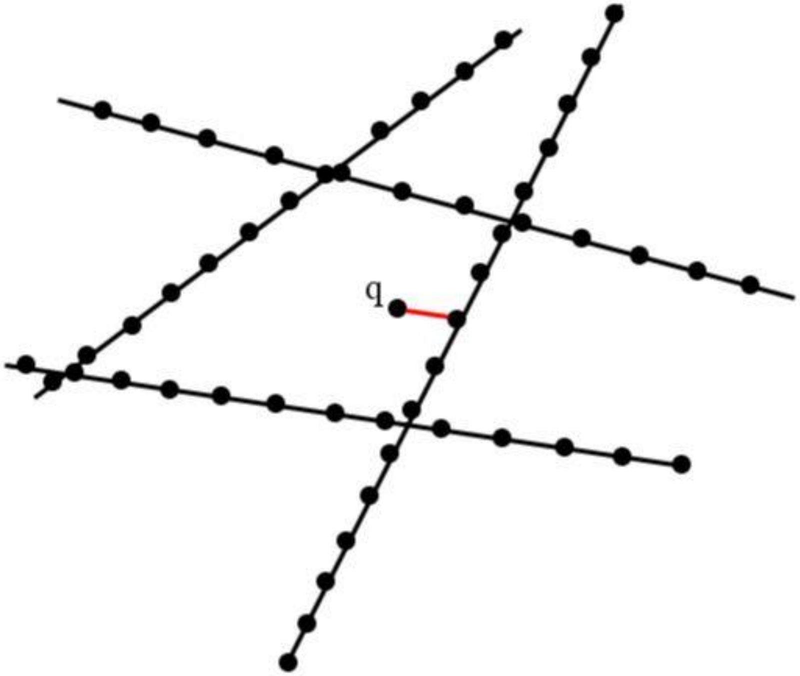
This helps us further restrict our search to almost parallel lines to ℓ_p

Net Module

- Intuition: sampling points from each line finely enough to get a set of points P, and building an $ANN(P, \epsilon)$ should suffice to find the approximate closest line.



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Lemma:

- Let x be the separation parameter: distance between two adjacent samples on a line
- Then
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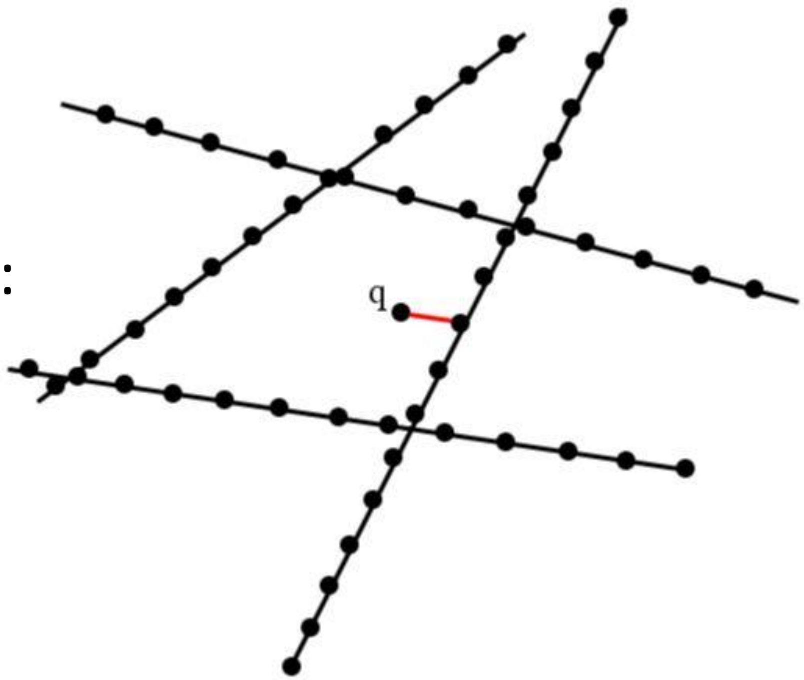
  - Or $dist(q, \ell_p) \leq x/\epsilon$



Parallel Module - Intuition

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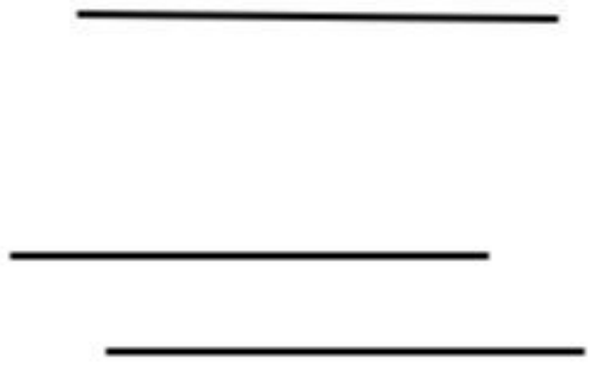

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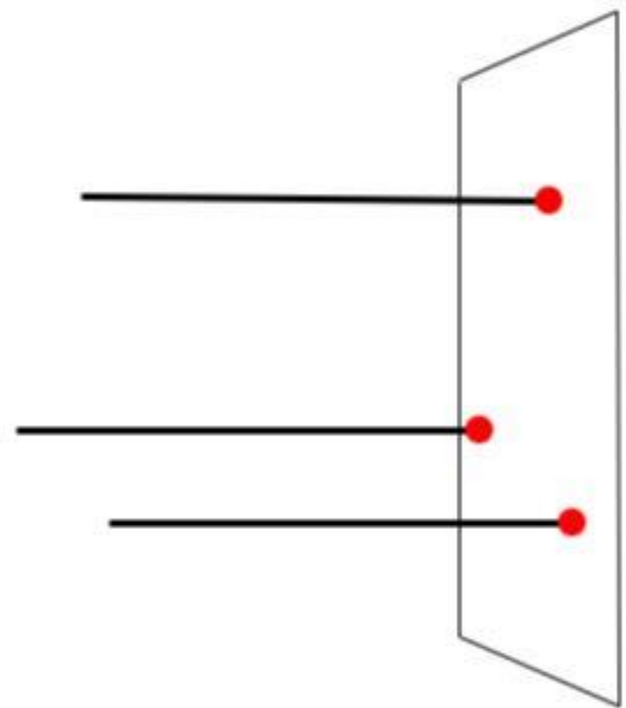

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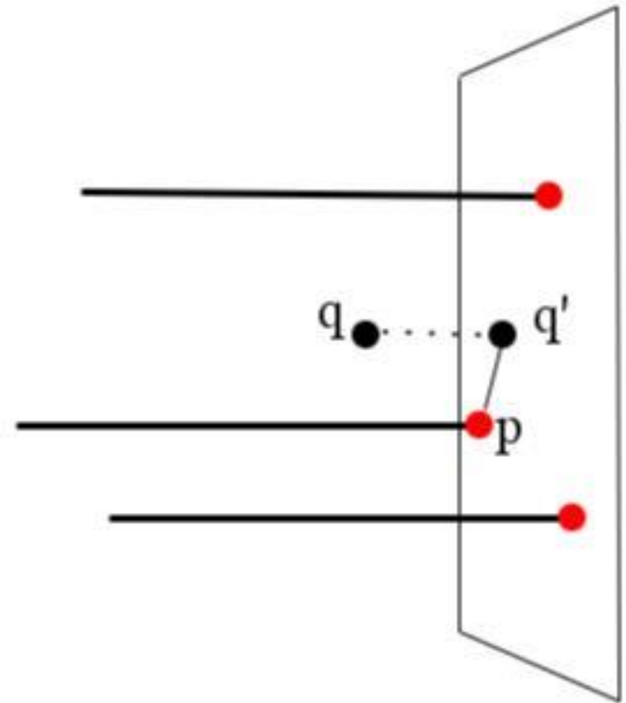
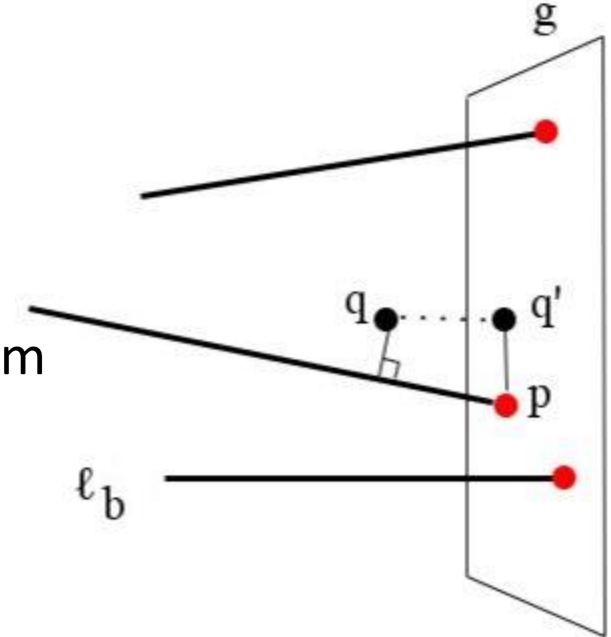


Parallel Module

- All lines in L are δ-close to a base line ℓ_b
- Project the lines onto a hyper-plane g which is perpendicular to ℓ_b
- Query is close enough to g
- Use the same data structure and query algorithm



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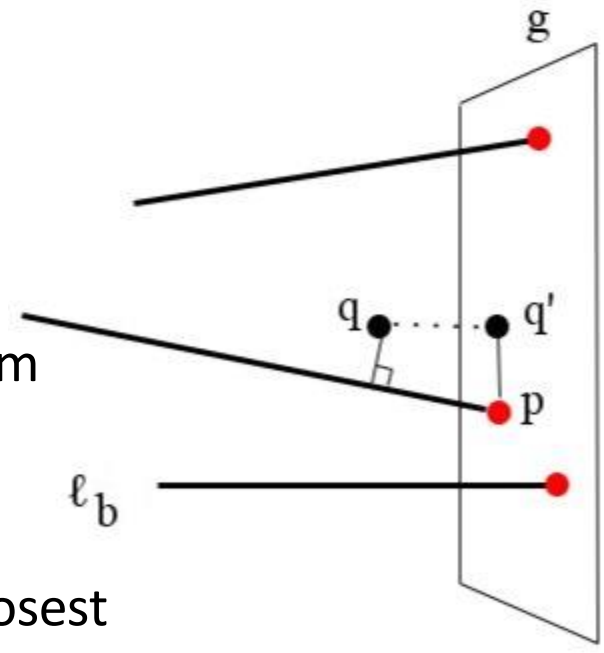

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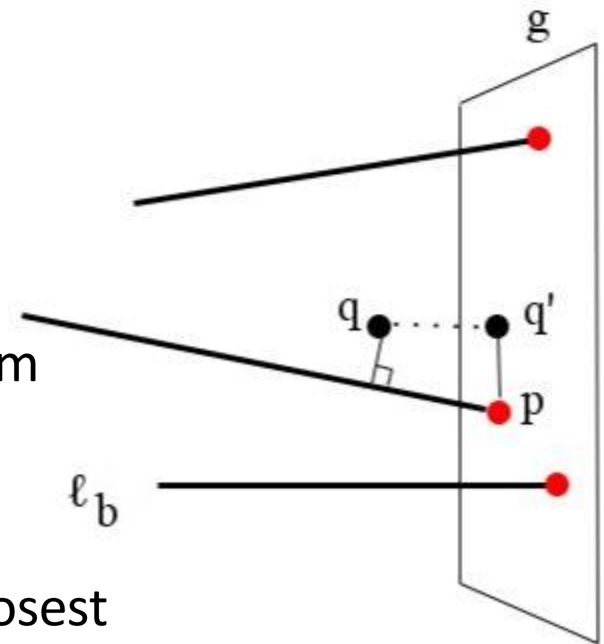

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Thus, for a set of almost parallel lines, we can use a set of parallel modules to cover a bounded region.

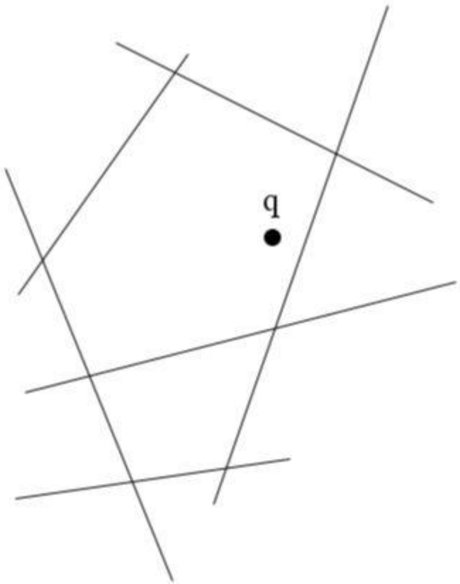
General Case

- Input lines can have any configuration
- Divergent Case
  - Input lines are $O(\epsilon)$-far from each other
- Almost Parallel Case
  - Input lines are all $O(\epsilon)$-close to each other

ALGORITHMS

Outline of the Algorithms

- **Input**: a set of n lines S



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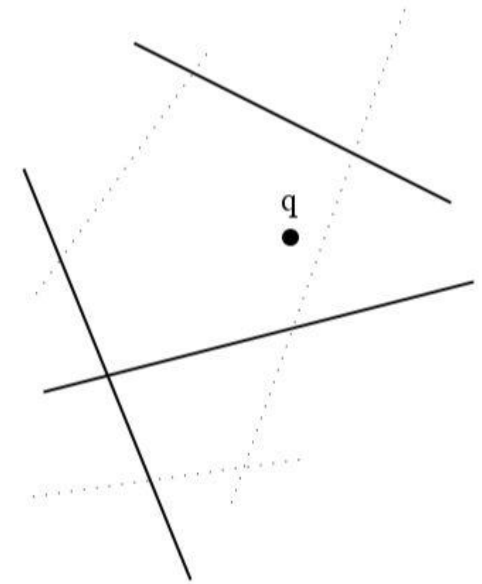

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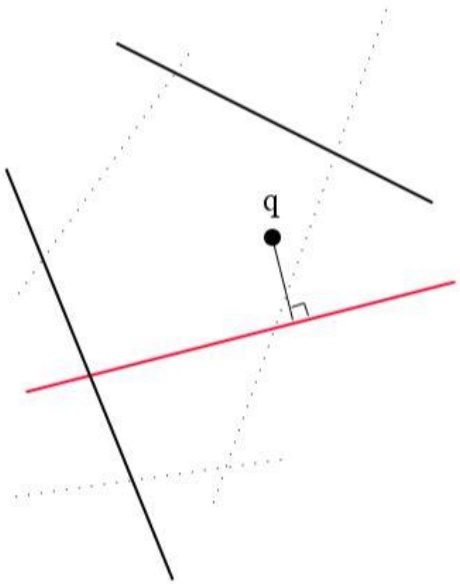

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- **Input**: a set of n lines S
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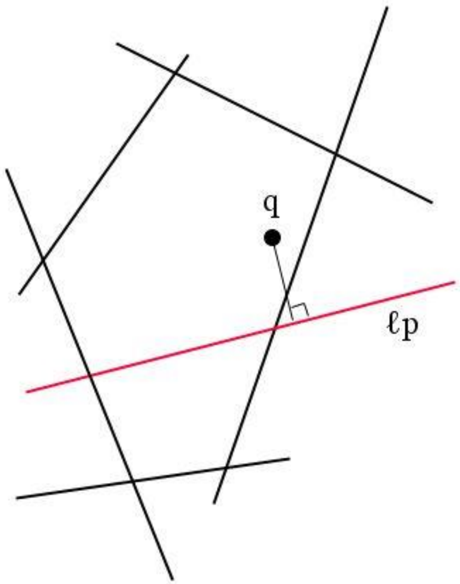
  - Use ℓ_p to find a much closer line ${\ell_p}'$
  - Update ℓ_p with ℓ_p'

Improvement step



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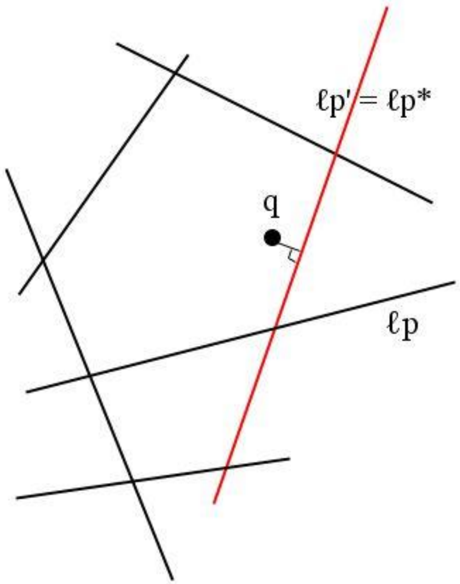

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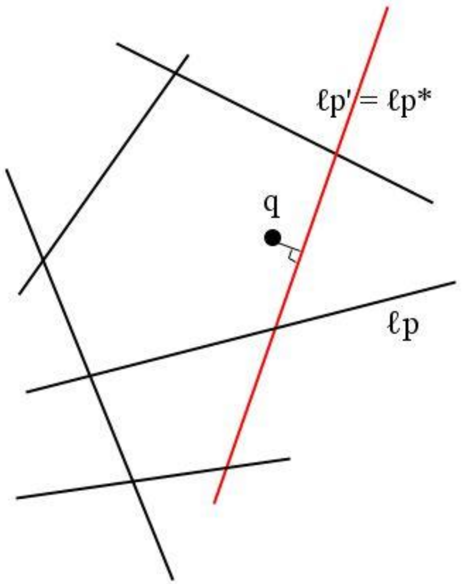

Why?

Outline of the Algorithms

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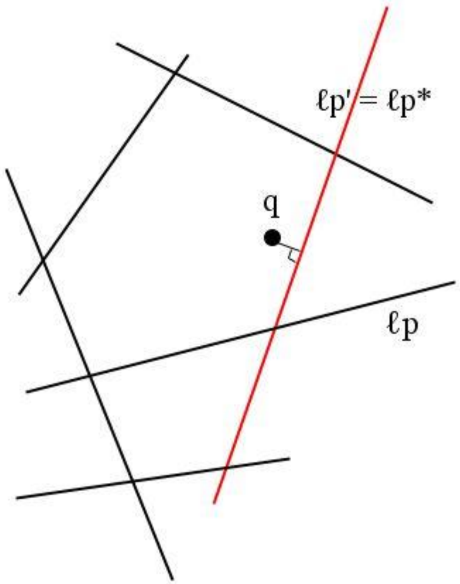

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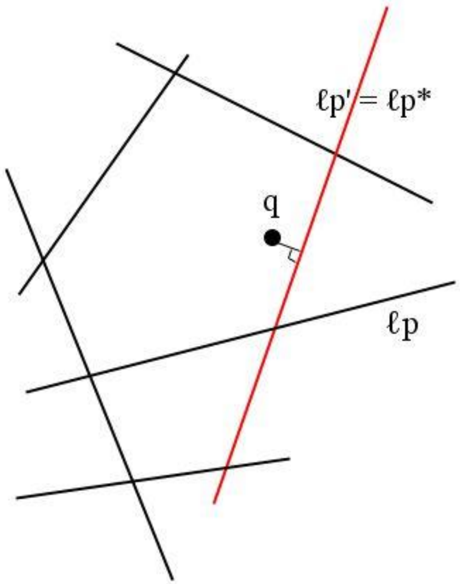


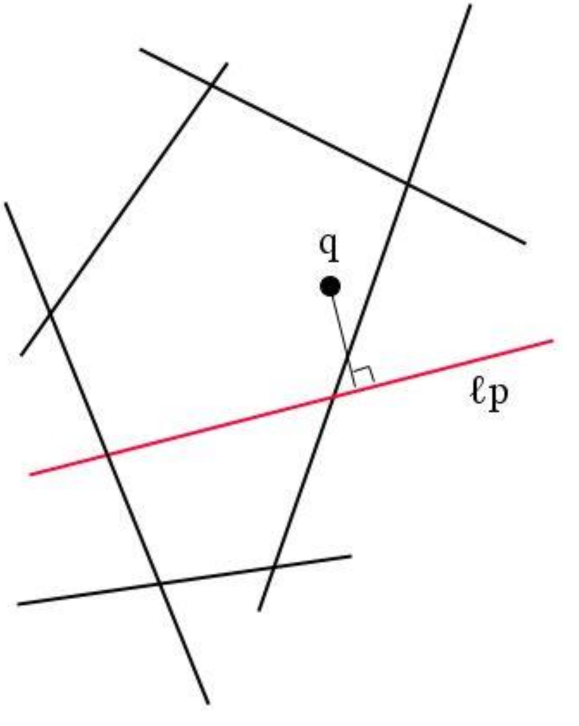
Let $\ell_1, \dots, \ell_{\log n}$ be the $\log n$ closest lines to q in the set S

With high probability at least one of $\{\ell_1, \dots, \ell_{\log n}\}$ are sampled in T

- $dist(q, \ell_p) \le dist(q, \ell_{\log n})(1+\epsilon)$
- $\log n$ improvement steps suffices to find an approximate closest line

Improvement Step

Given a line ℓ, how to improve it, i.e., find a closer line?



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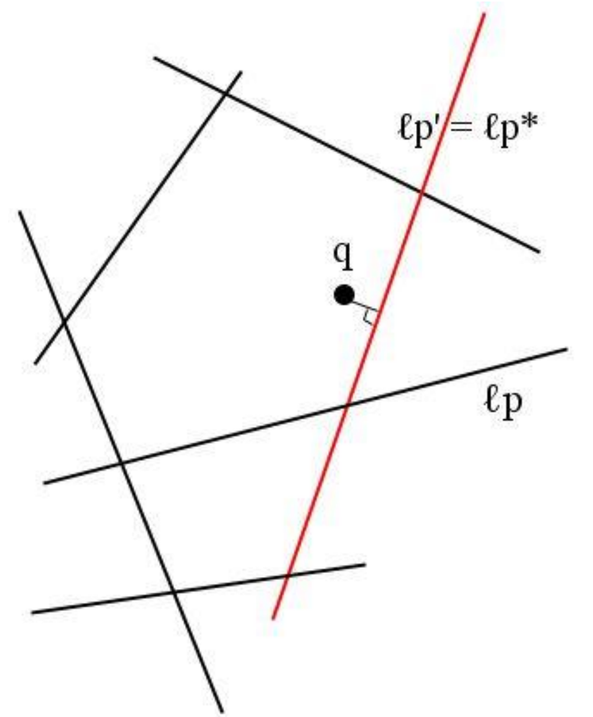
- Data structure
- Query Processing Algorithm



General Case

- Search among all lines that are ϵ-far from current line using Divergent Case

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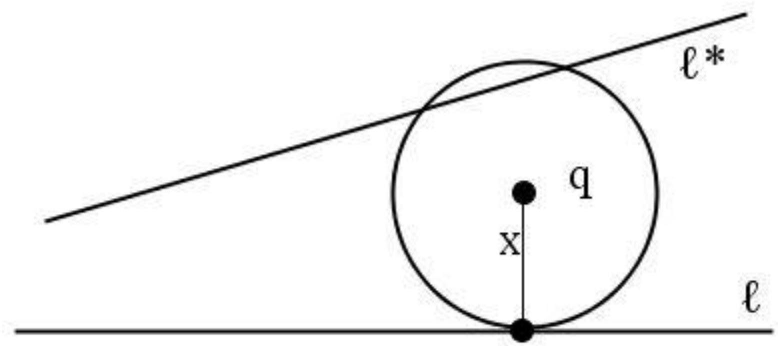
- Search among all lines that are ϵ-far from current line using Divergent Case
- Search among the lines that are almost parallel to line found in previous step using Almost Parallel Case

Divergent Case

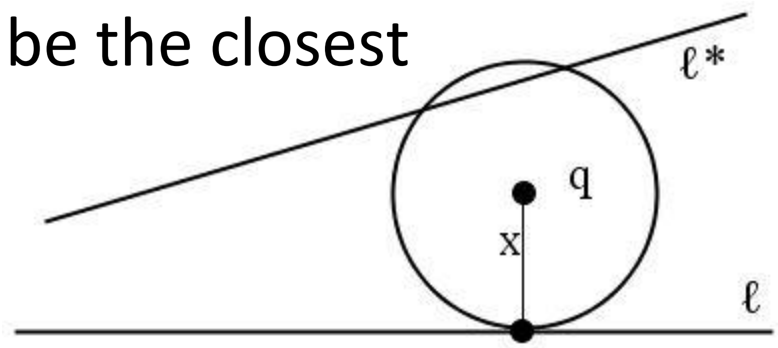
Assume any two lines are ϵ-far; they diverge quickly.



Divergent Case

Assume any two lines are ϵ-far; they diverge quickly.

- Let ℓ be the current line, and ℓ^* be the closest line to q
- Let $x = dist(q, \ell)$
- $dist(q, \ell^*) \leq x$



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- Let ℓ be the current line, and ℓ^* be the closest line to q
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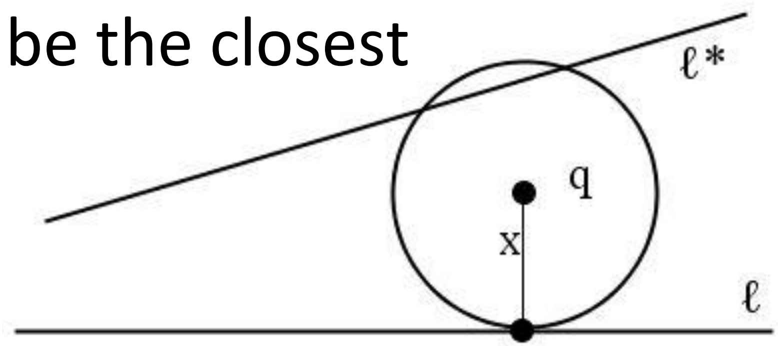
  - All potential ℓ^* intersect $B(q, x)$



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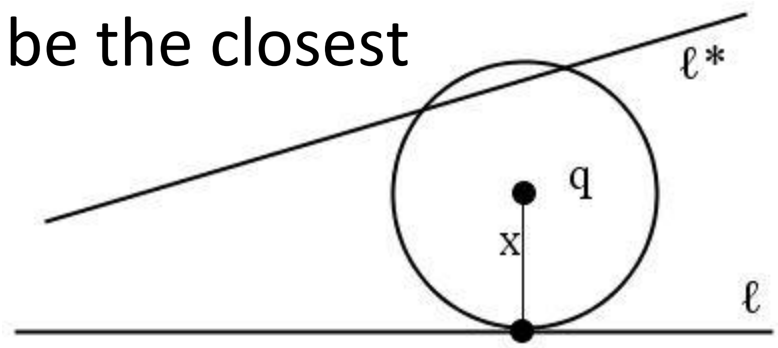

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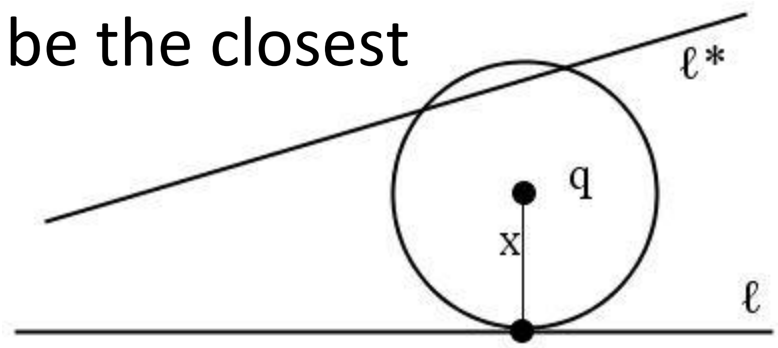

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  - All potential ℓ^* intersect $B(q, x)$
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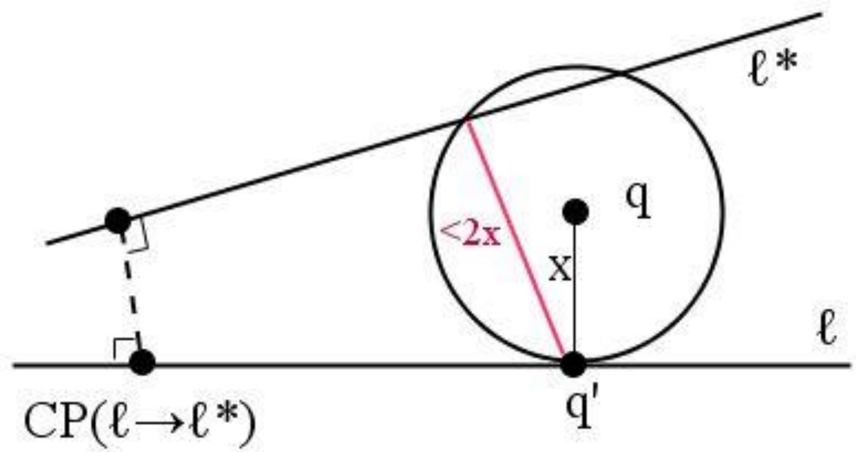
  - Bad news: we don't know this ball in advance

Divergent Case contd.

What we know:

- $dist(\ell, \ell^*) \leq 2x$
- Let q' be the projection of q on ℓ



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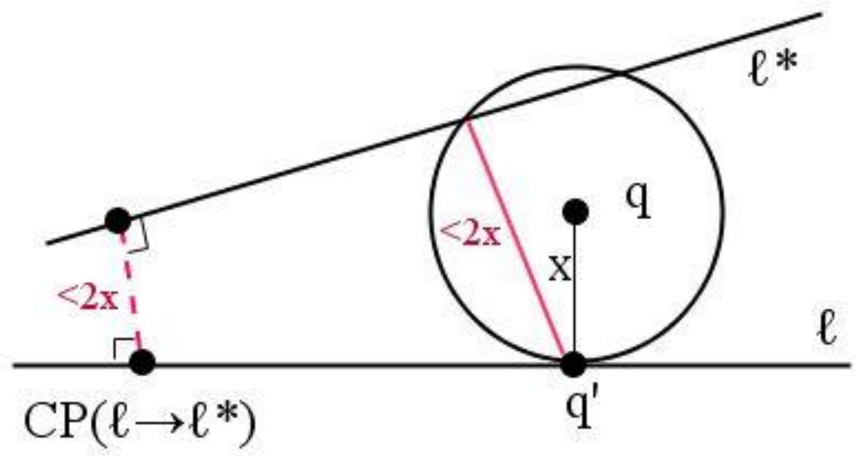

Divergent Case contd.

What we know:

- $dist(\ell, \ell^*) \leq 2x$
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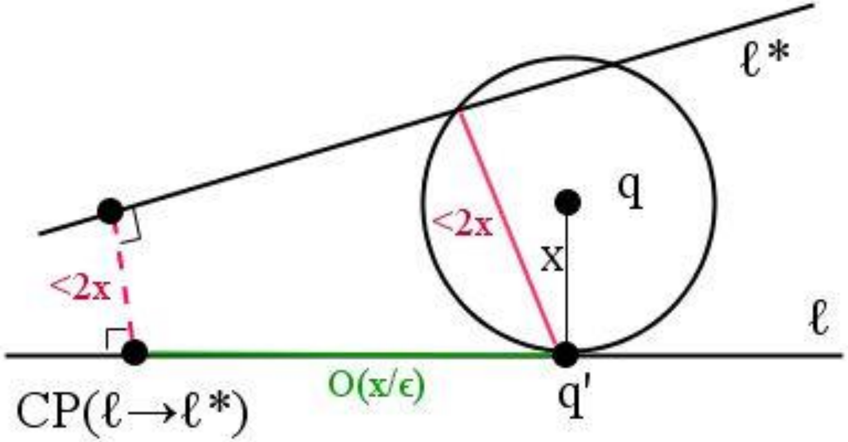
  - $CP_{\ell \to \ell^*}$ is not farther than $\frac{x}{\epsilon}$ from q' since they are ϵ-far



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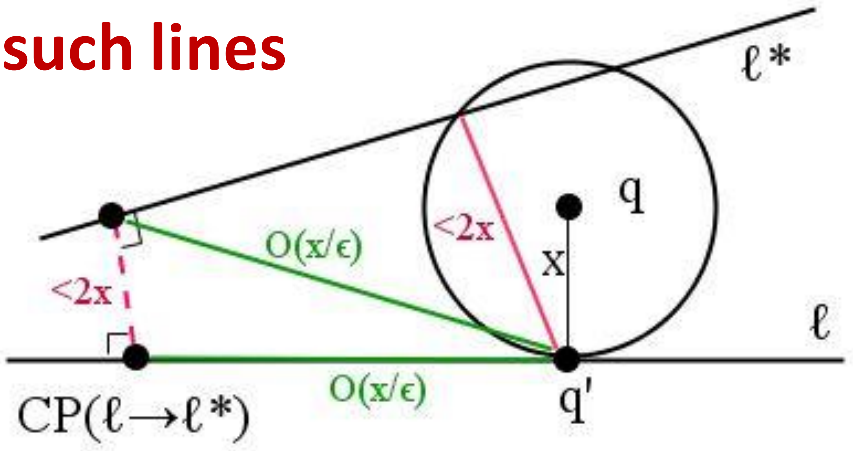
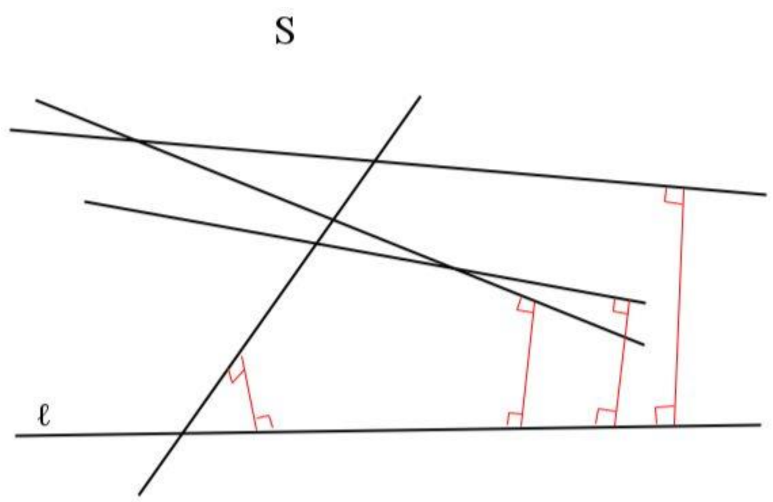
  - $\boldsymbol{B}(\boldsymbol{q}', \boldsymbol{O}\left(\frac{\boldsymbol{x}}{\boldsymbol{\epsilon}}\right))$ **touches all such lines**



Data Structure

For each $\ell \in S$

- Sort all lines ℓ' according to their distance from ℓ



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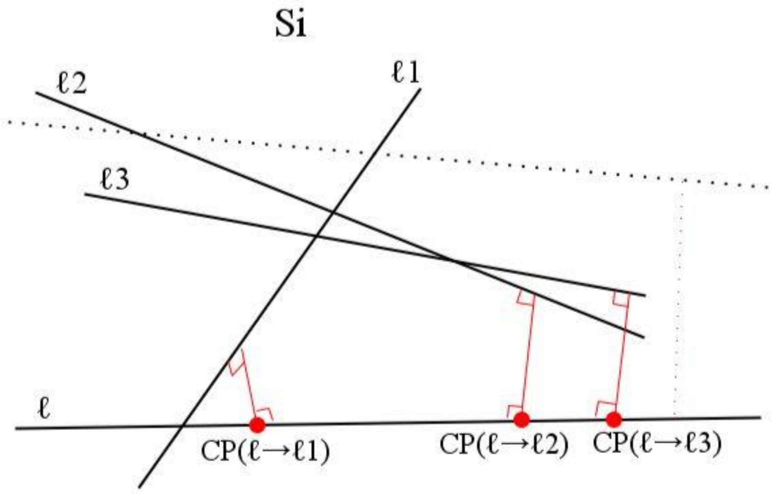

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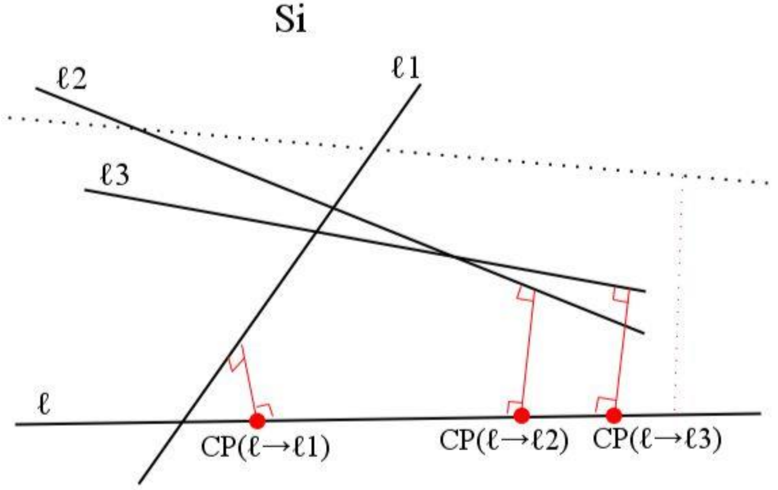

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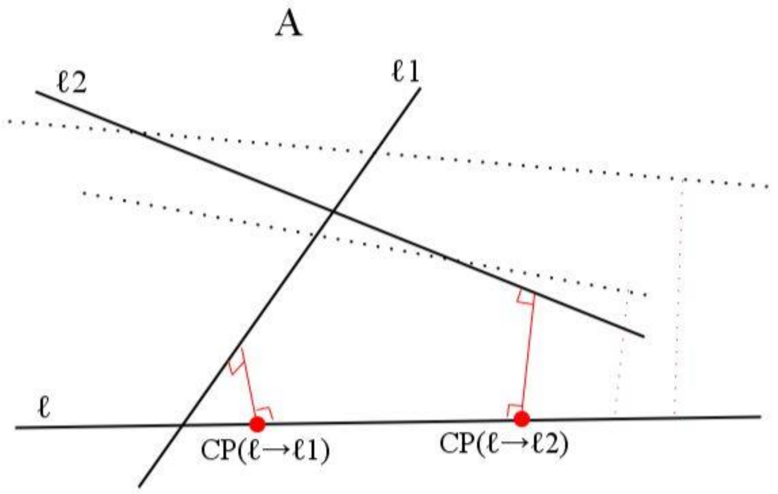

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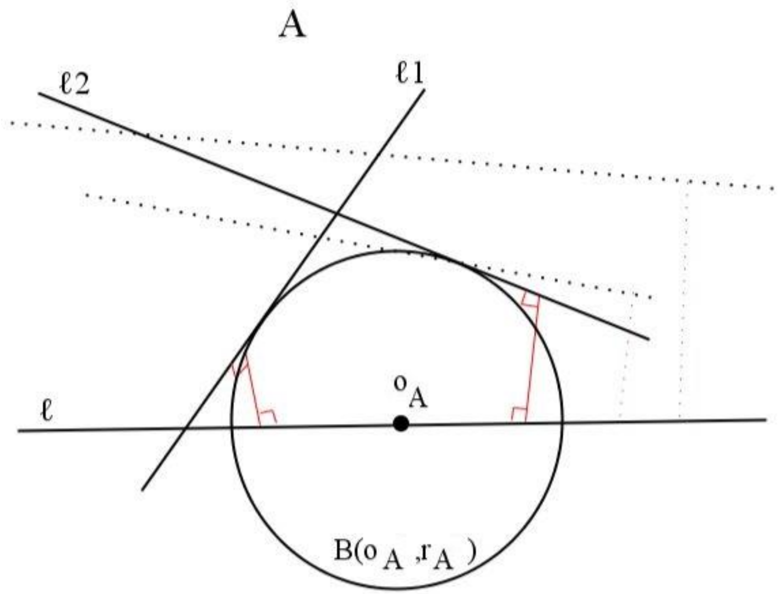
    - Find smallest ball $B_A(\mathrm{o_A}, \mathrm{r_A})$ with its center on ℓ which intersects all lines in A
      -> ($r_A \leq O(\frac{x}{\epsilon})$)



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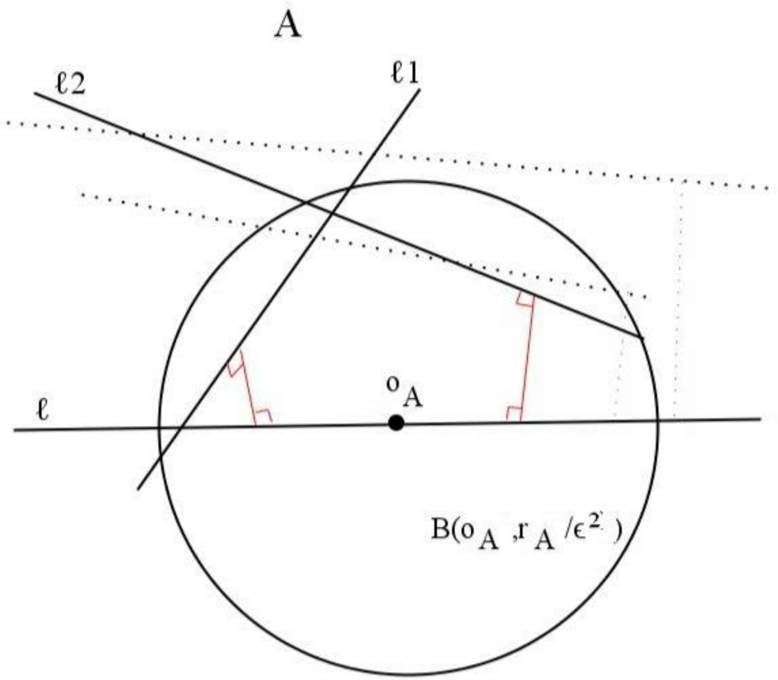
(#samples = O$(n\ r_A/(\epsilon^2 r_A \epsilon^3)) = O(n/\epsilon^5))$



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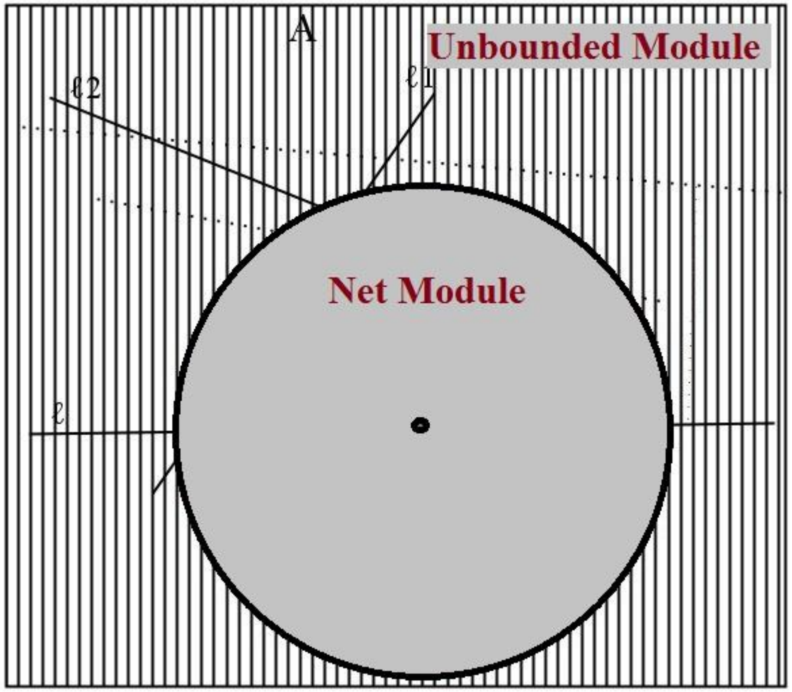
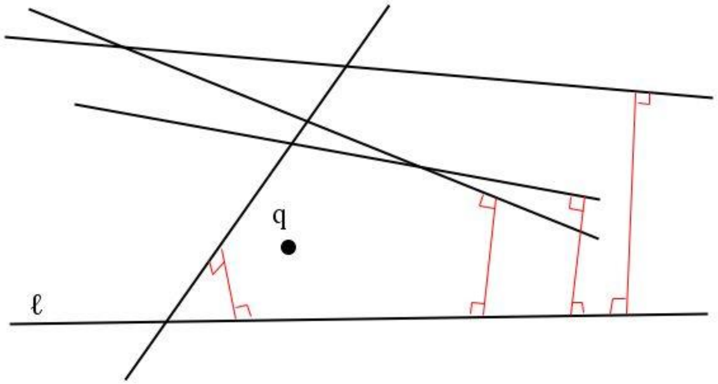


Query Processing Algorithm

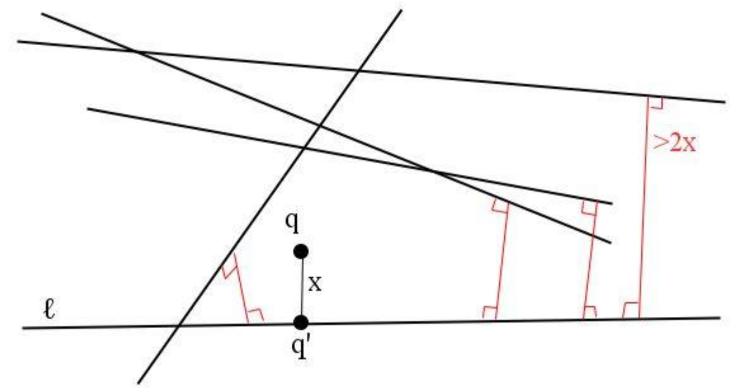
Given query point q



Query Processing Algorithm

Given query point q

- Project q on ℓ to get q'
- Use binary search to find the set A of all lines ℓ' that are within distance $2x$ of ℓ, and that $CP_{\ell \to \ell'}$ is within distance $2x/\epsilon$ of q'



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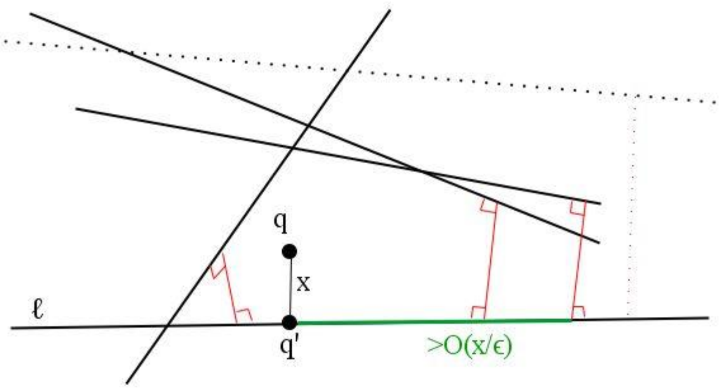

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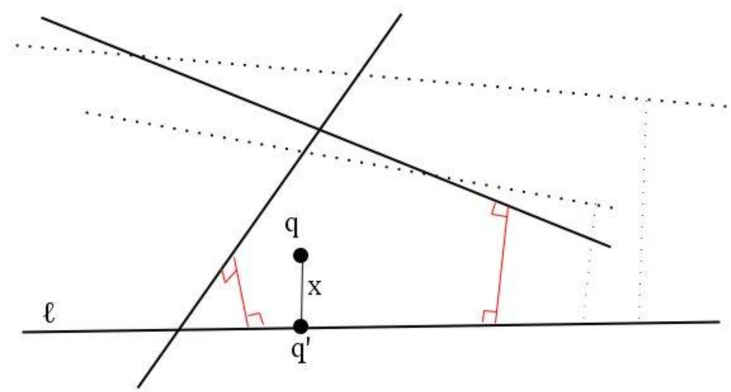

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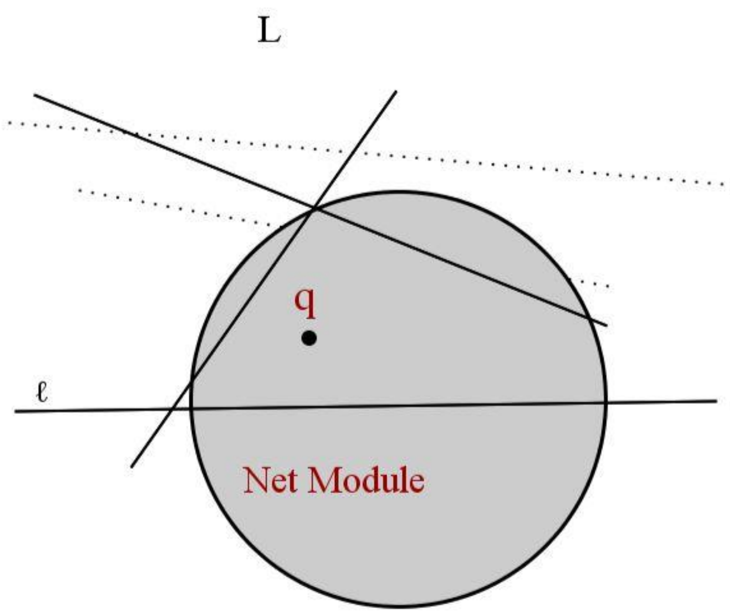

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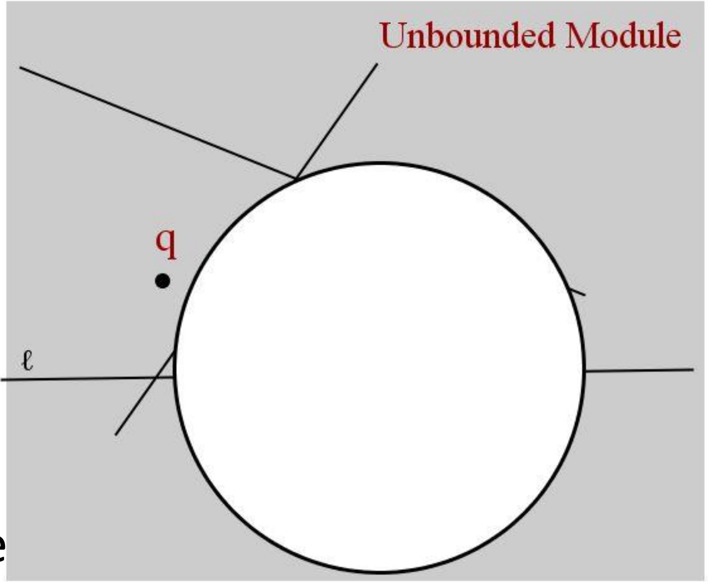
- Otherwise use unbounded module to find the approximate closest line -> done!

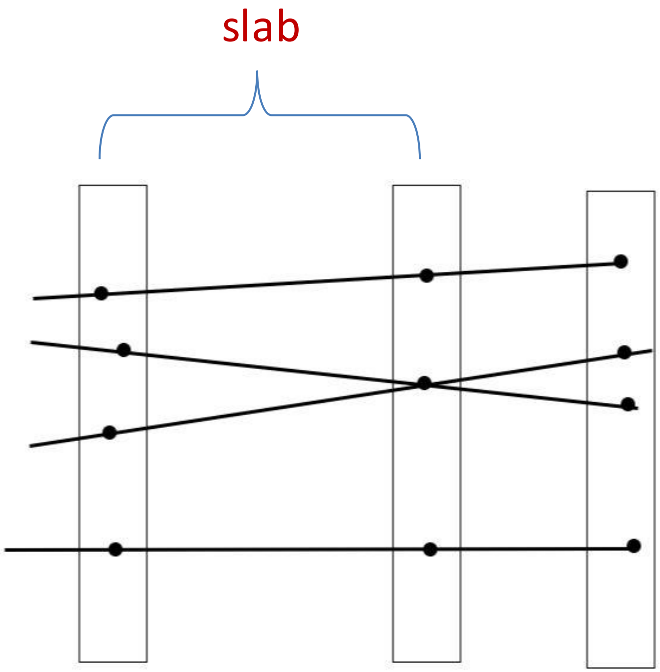


Almost Parallel

All lines are 2ϵ-close to each other.

For each line ℓ

- Partition the space into slabs using perpendicular hyperplanes to ℓ s.t. for any pair of lines ℓ_1, ℓ_2:



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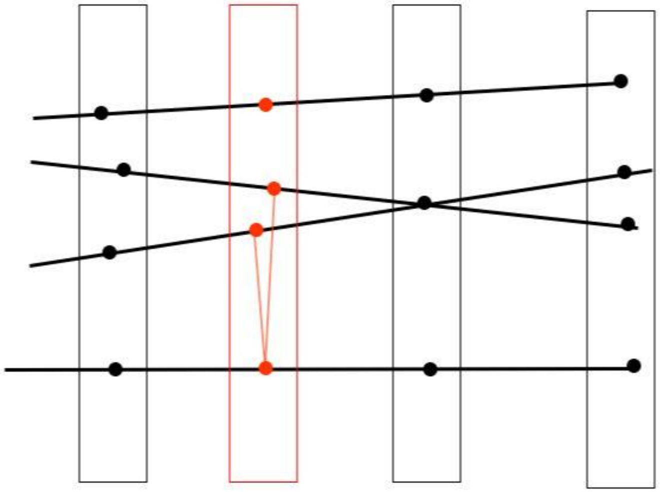
There is a unique ordering of the lines



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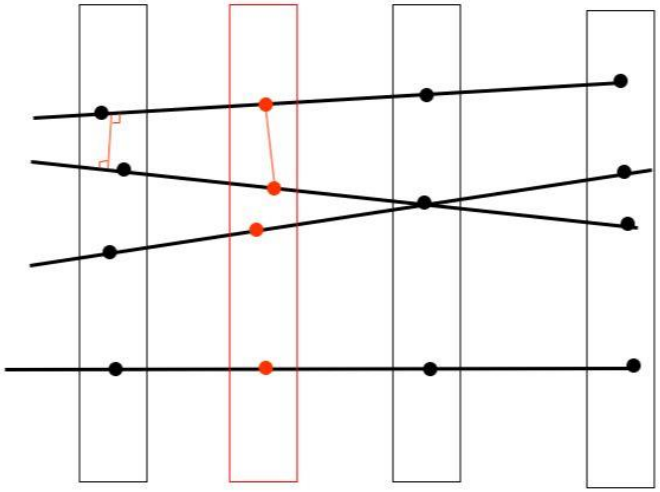
  - $dist_{H(\ell,o)}(\ell_1, \ell_2)$ on the hyper-plane is monotone



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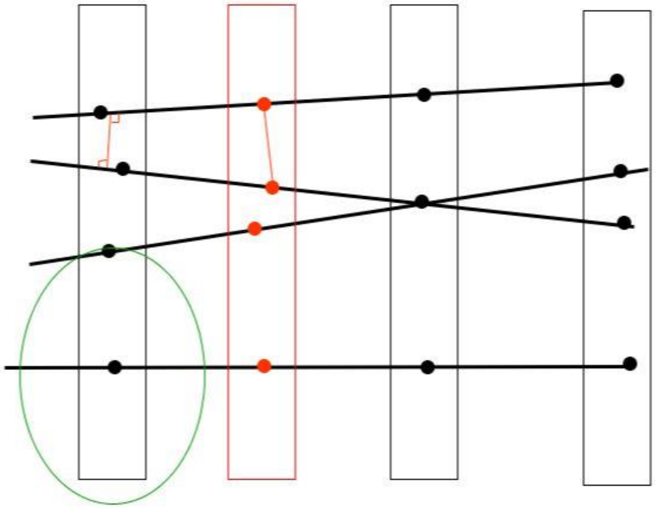
  The minimum ball intersecting any prefix of lines have its center on the boundary of slab



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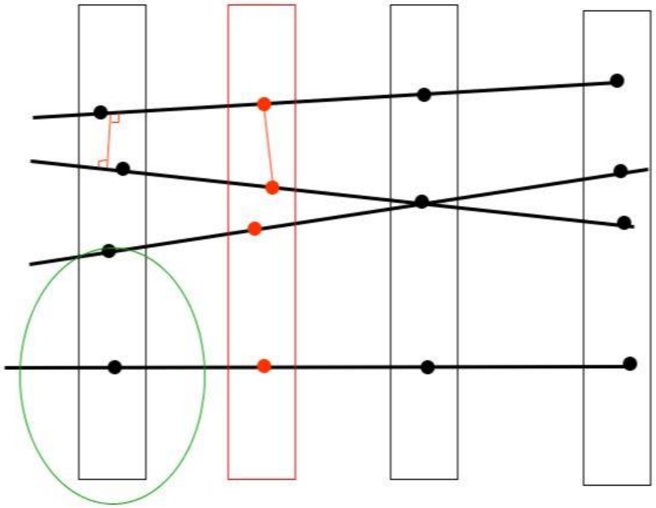
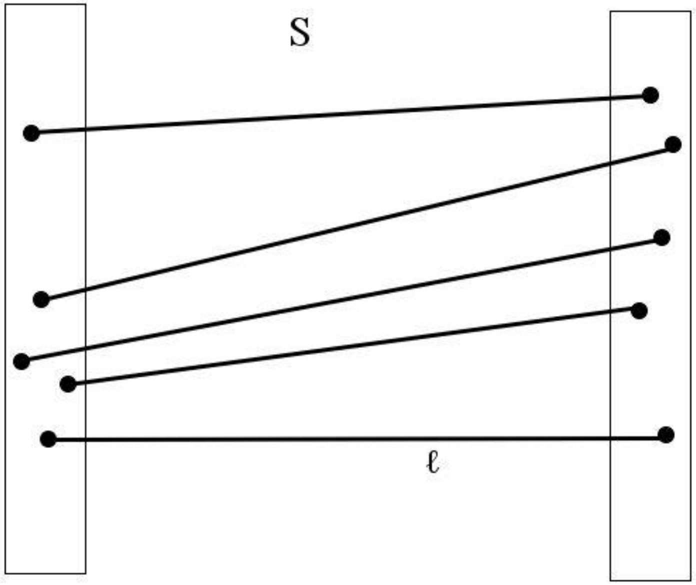
  - $dist_{H(\ell,o)}(\ell_1, \ell_2)$ on the hyper-plane is monotone

  The minimum ball intersecting any prefix of lines have its center on the boundary of slab.
- $O(n^2)$ slabs suffices



Data Structure in Each Slab

- For each i, let $B(o, r)$ be the smallest ball touching the closest i^{th} lines s.t. $o \in \ell$. We know o would be on the boundary of slab.



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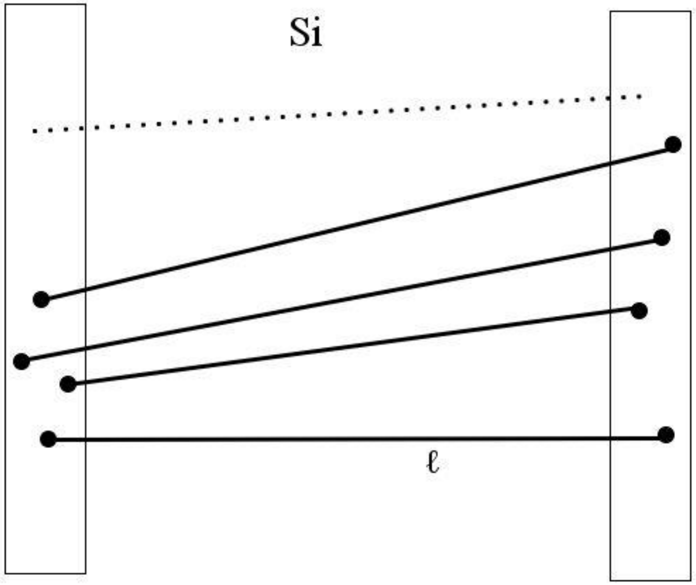

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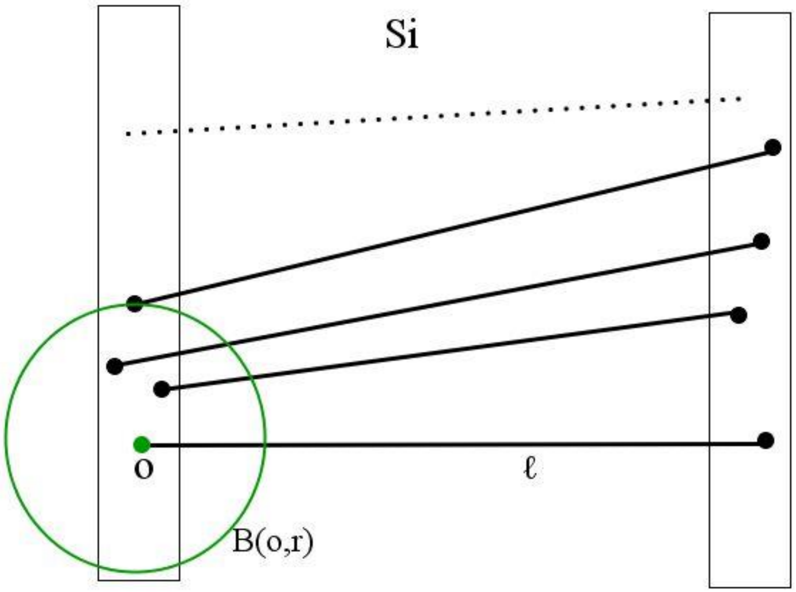

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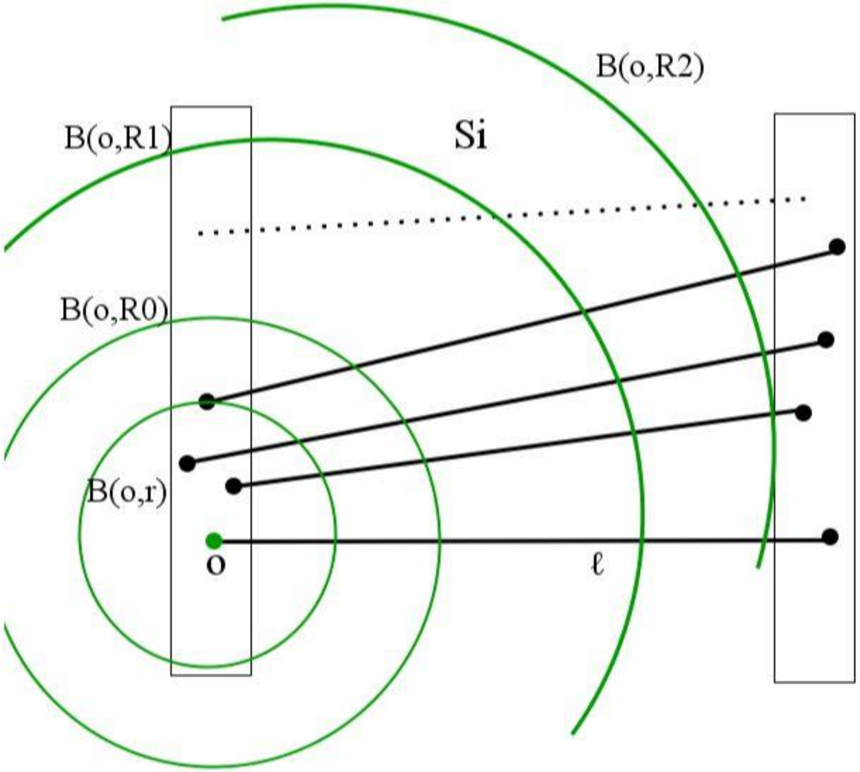
- Let $\delta_0 > \cdots > \delta_t$ be all pairwise angles
- Let $R_0 = \frac{r}{\epsilon \delta_0}, \ldots, R_t = \frac{r}{\epsilon \delta_t}$
- Consider the balls $B(o, R_0), \ldots, B(o, R_t)$



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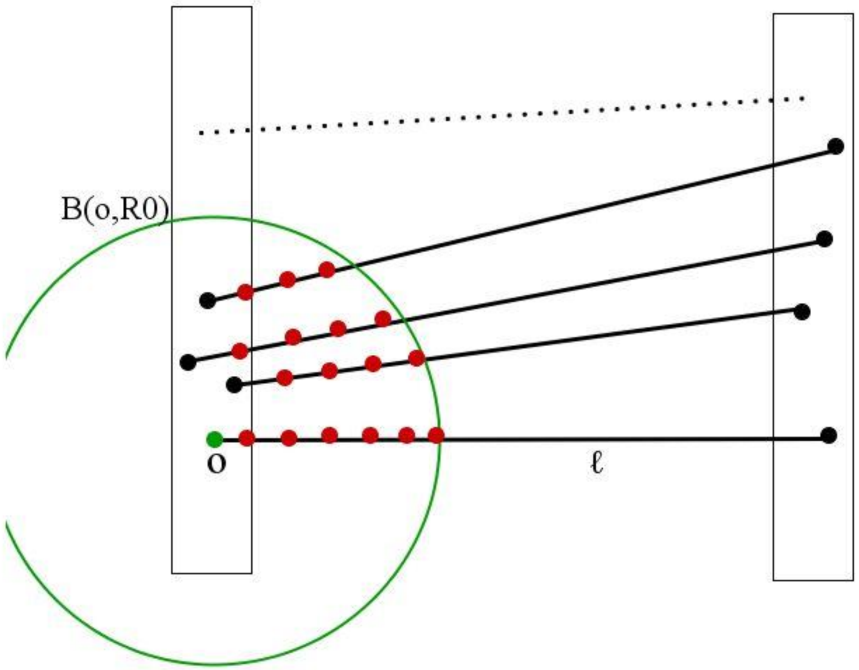

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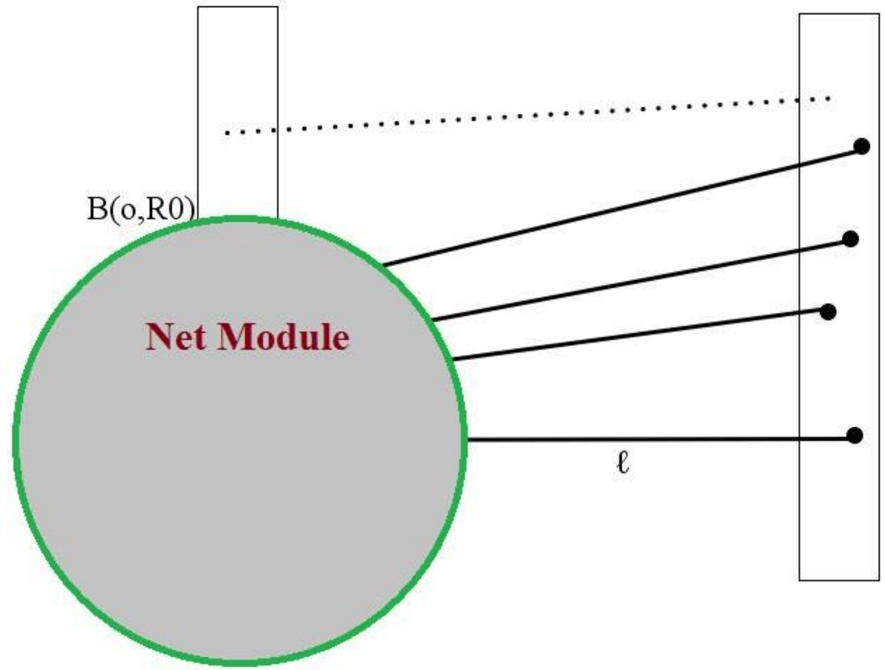

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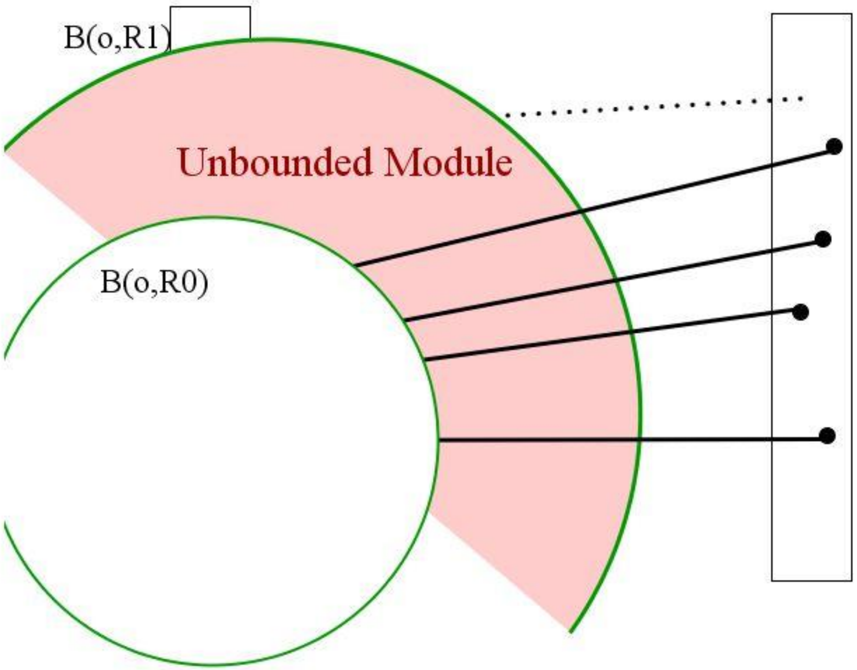
- For each ball $B(o, R_i)$
  - Build unbounded module on it



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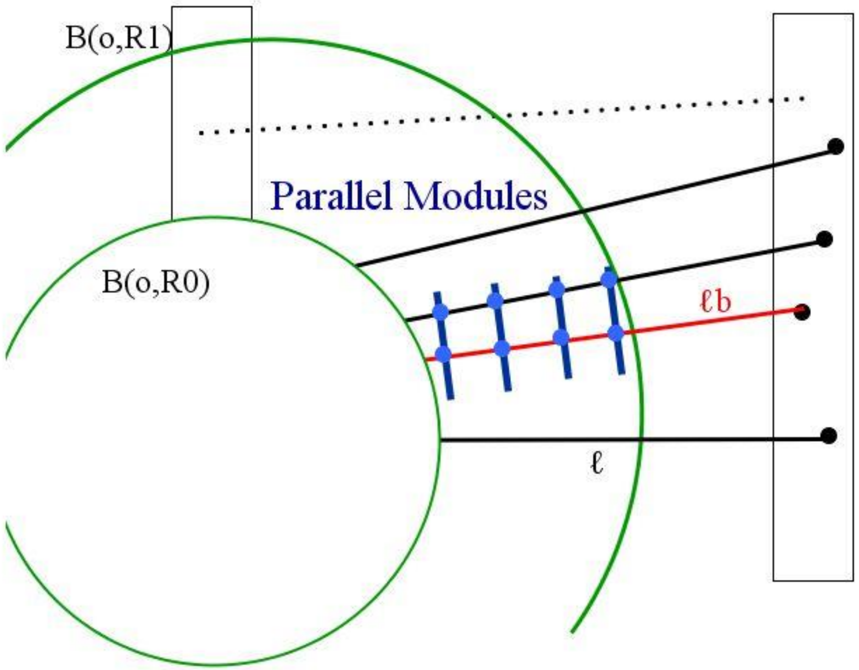
  - For each line ℓ_b
    - Build a set of parallel modules with ℓ_b as their base line for all the lines that are δ_i-close to ℓ_b , so that they cover the space between $B(o, R_i)$ and $B(o, R_{i+1})$ with separation $R_{i+1}\epsilon$



Query Processing Algorithm

- Given q, find the right slab, and retrieve all candidate lines
- Using binary search find r

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- Given q, find the right slab, and retrieve all candidate lines
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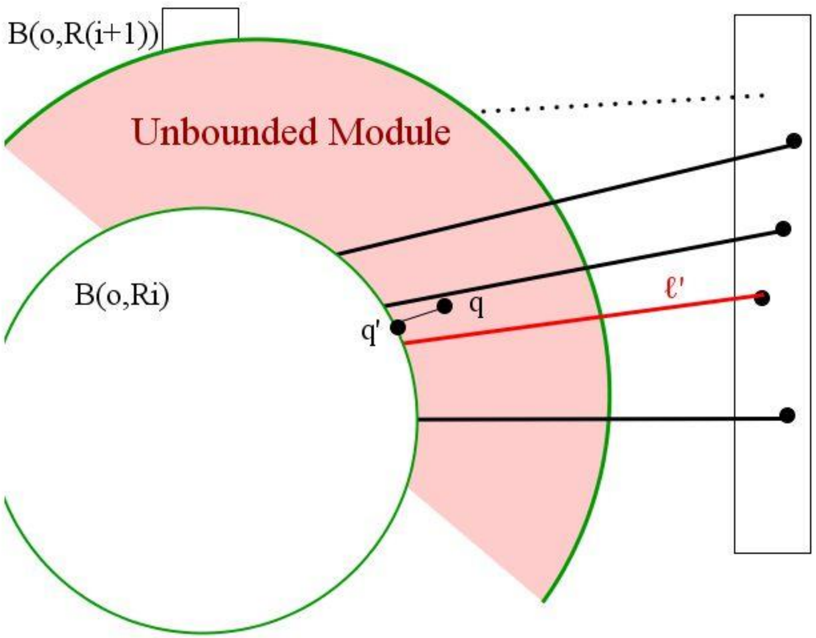

Query Processing Algorithm

- Given q, find the right slab, and retrieve all candidate lines
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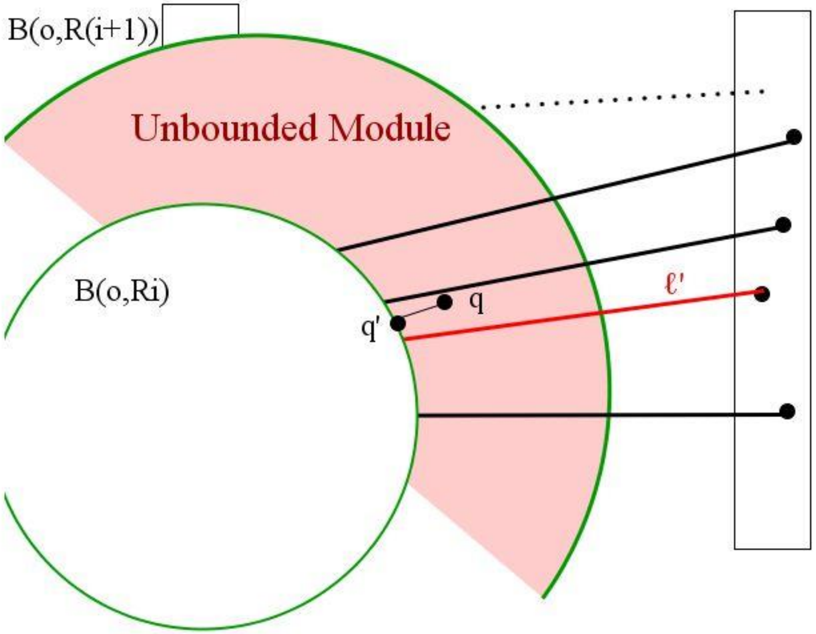

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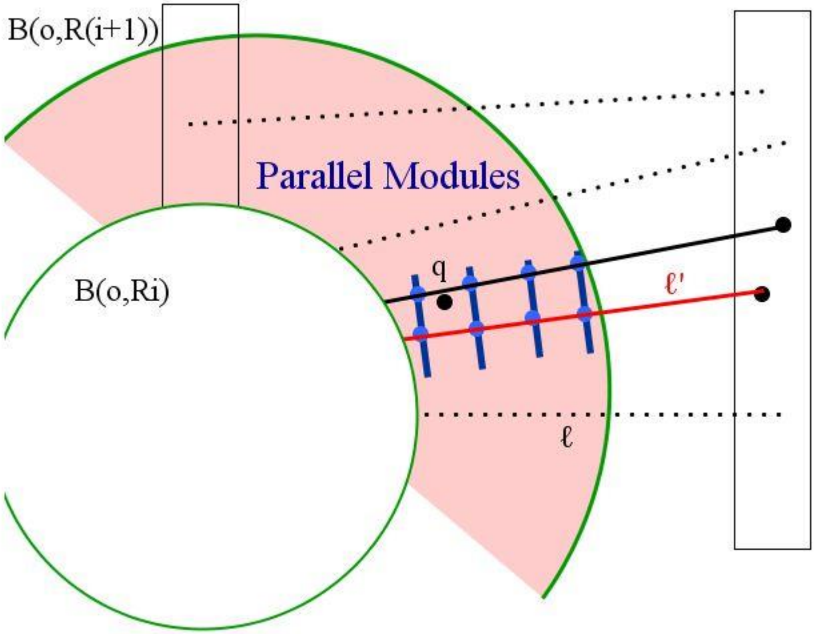

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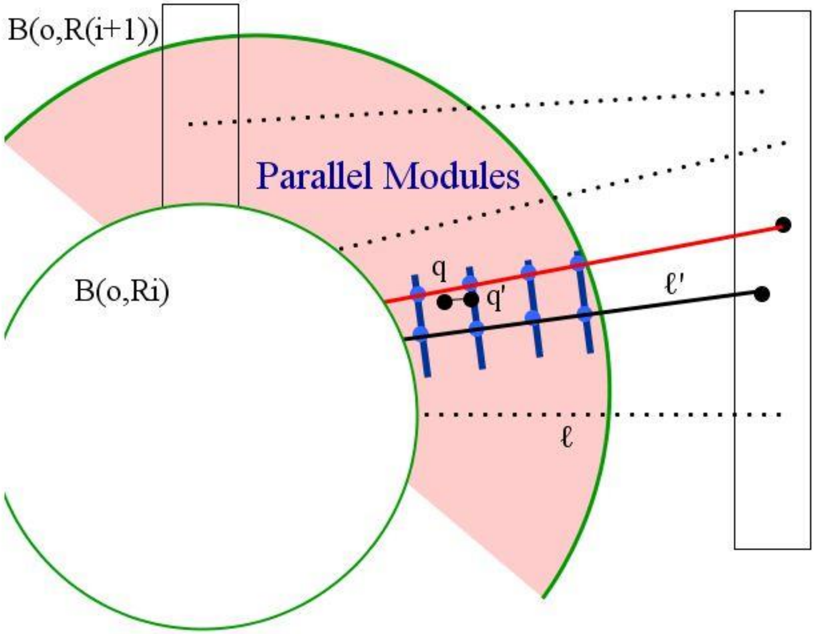
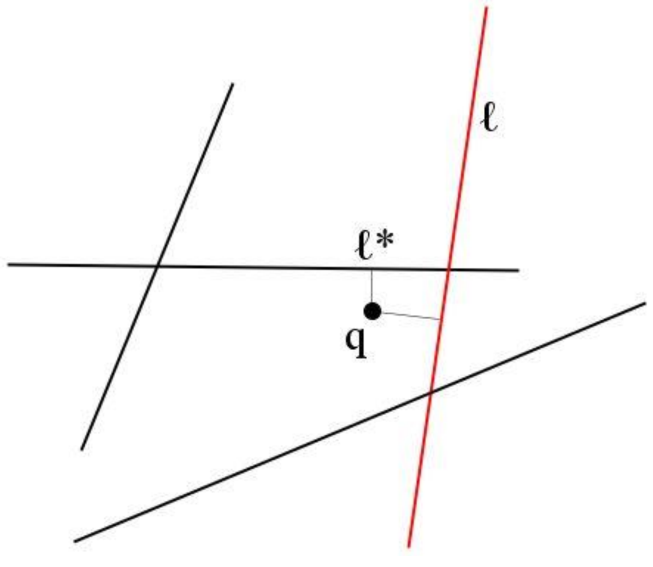


Summary

- Nearest Line Search Problem



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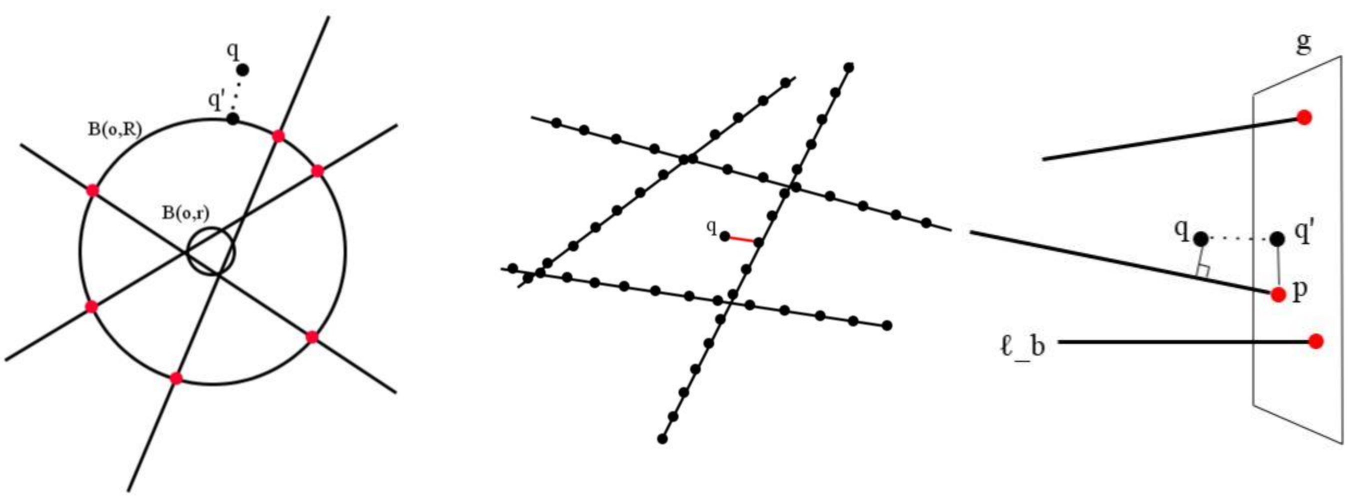
- Modules: unbounded, net, parallel



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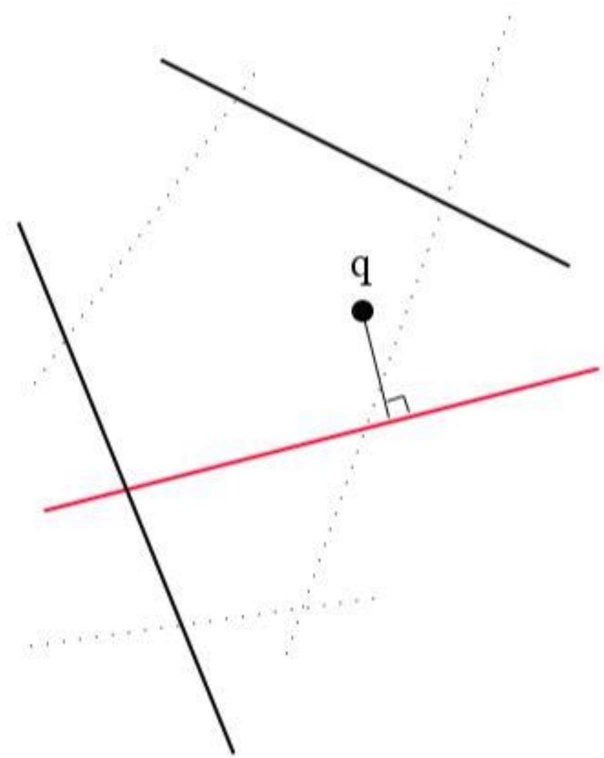

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- Nearest Line Search Problem
- Modules: unbounded, net, parallel
- Use of random sampling
- How to improve given a line
- Bounds of our algorithm
  - Polynomial Space:
    $$\left(\frac{dN}{\epsilon}\right)^{O(1)} \times \mathcal{S}\left(\left(\frac{N}{\epsilon}\right)^{O(1)}, \epsilon\right) = O(N+d)^{O\left(\frac{1}{\epsilon^2}\right)}$$
  - Poly-logarithmic query time :
    $$(d \log N)^{O(1)} \times \mathcal{T}(\left(\frac{N}{\epsilon}\right)^{O(1)}, \epsilon) = \left(d + \log N + \frac{1}{\epsilon}\right)^{O(1)}$$

Future Work

- The current result is not good in practice
  - Large exponents
  - Algorithm is complicated

  Can we get a simpler algorithms?

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  Can we get a simpler algorithms?
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- Generalization to other objects, e.g. balls

THANK YOU!